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Analytical Investigation of Interactions between a Flexible Circular Plate with Transversely Isotropic Half-space

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1- Introduction

Because of both engineering applications and mathematical complexities, contact problems are of interest to both engineers and mathematicians. Applications range from foundation design, pavement engineering, deflectometer tests and soil-structure interaction to electronic components technology, mechanics of adhesives and thin films. Depending on the relative stiffness of the materials in contact, geometry of contact area, and load configurations, many of the underlying mixed boundary value problems are nonlinear. From rigid-elastic contact to elastic-elastic contact we may pass from singular to regular contact stress distributions. A central boundary value problem on this subject is the tensionless contact problem of a loaded flexible plate rested on elastic materials.

In this paper, a flexible circular plate as foundation attached to the surface of a vertically symmetric transversely isotropic homogenous linear elastic half-space are considered to be under the effect of an arbitrary axisymmetric force. The contact area of the foundation and the half-space is considered to be both frictionless and tensionless. Since the contact area is tensionless and may transfer only pressure, the boundary value problem is a nonlinear one, and the contact region is determined with the use of a trial and error procedure. To do so, ring load Green's functions for both the plate and the half-space are determined and the continuity condition of displacements at the contact area is satisfied with the use of an integral equation that has to be solved by satisfying several inequalities. The inequalities are defined to model the tensionless contact at the contact region. The set of integral equations and the inequalities are solved with the use of ring-shape finite element method with different sizes, which lets us capture the singularity at the edge of some relative rigid plates and the regularity at any other cases. The validity of the hybrid form of analytical and numerical methods is illustrated by comparing the results of this paper with a number of different cases of both linear and nonlinear interactions of a circular plate and half-space, which have been previously reported. The results are beneficial for a better understanding of the mechanics of similar boundary value problems and to have optimum design of foundations.

2- Mathematical Formulation

A half-space containing linear elastic transversely isotropic materials is considered to be under the effect of a flexible surface annular plate with inner radius a_0 and outer radius a (Fig 1.). If one sets $a_0 = 0$, then the plate will be of a disc shape, which may be called a solid plate. The annulus is assumed to be under an axisymmetric vertical load distribution $q_z(r)$, which makes a contact pressure $p_z(r)$ in between the annulus and the half-space (see Fig 2.). The contact area is assumed to be tensionless, and thus the contact region may transfer only pressure. The radius α is used to show the upper limit of the contact region. We also assume that the plate is governed by the classical Kirchhoff theory, with E_p as Young's modulus, and ν_p as Possion's ratio. A continuum theory is considered for the half-space containing transversely isotropic materials.

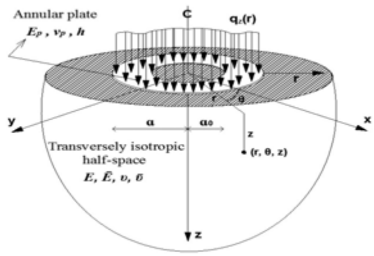

Fig 1. A flexible annular plate on a transversely isotropic half-space.

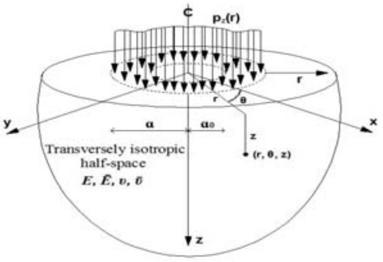

Fig 2. A transversely isotropic half-space under the contact pressure.

2.1- Ring load Green's functions for plates

Assuming small elastic deformations for an isotropic solid circular plate under an axisymmetric distributed loading denoted as $q_z(r)$ and support reaction $p_z(r)$, the response of classical governing equation for the vertical deflection $w_p(r)$ in a cylindrical coordinate system is:

$$w_p(r) = w_p(0) + \int_0^a \bar{w}_p^{\,solid}(r;r')\left[q_z(r') - p_z(r')\right]dr', \qquad 0 < r < a$$

Where

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$$\bar{w}_p^{\,solid}(r;r') = \left(\frac{2a^2\left(\ln r'(\nu_p+1)+\nu_p+1\right)+r'^2(1-\nu_p)}{16\pi D a^2(\nu_p+1)} r^2 - \frac{r^2 \ln r}{8\pi D} \right) \times H(r'-r)$$
$$+\left(\frac{r'^2}{8\pi D}\ln r - \frac{r'^2(\nu_p-1)}{16\pi D a^2(\nu_p+1)} r^2 - \frac{r'^2(\ln r'-1)}{8\pi D} \right) \times H(r-r').$$

2.2- Half-space ring-load Green's functions

A cylindrical coordinate system $\{(o:r,\theta,z), z>0\}$ is attached to the surface of a transversely isotropic linear elastic half-space whose material axis of symmetry is taken to be parallel to the z-axis, which itself is depth-wise. The displacements and stresses at any point of the half-space under the load due to the plate can be expressed by the integral equation

$$u_h(r,z) = \int_0^{\alpha} \bar{u}_h(r,z;r')\, p_z(r')\, dr'$$

$$w_h(r,z) = \int_0^{\alpha} \bar{w}_h(r,z;r')\, p_z(r')\, dr'$$

$$\sigma_{zzh}(r,z) = \int_0^{\alpha} \bar{\sigma}_{zzh}(r,z;r')\, p_z(r')\, dr'$$

$$\sigma_{rzh}(r,z) = \int_0^{\alpha} \bar{\sigma}_{rzh}\ (r,z;r')\, p_z(r')\, dr'$$

2.3- Tensionless plate governing equations

The axisymmetric tensionless and frictionless contact problem for a solid plate may be formulated by assuming a single continuous contact area of radius $\alpha \le a$. The vertical displacement compatibility and tensionless conditions over the contact area should be enforced in between the half-space and the plate, which leads to the following integral equation

$$\int_0^{\alpha} \left[\bar{w}_h(r,0;r') + \bar{w}_p(r;r') \right] p_z(r')\, dr' - w_p(0)$$
$$= \int_0^{a} \bar{w}_p(r;r')\, q_z(r')\, dr'. \quad 0 \le r < \alpha$$

$$F_z = 2\pi \int_0^{a} r' q_z(r')\, dr' = 2\pi \int_0^{\alpha} r' p_z(r')\, dr'$$

3- Ring element method

With the use of this method, the displacement at each point of the medium, say $M(r,\theta,z)$, is written as

$$W(M) = \sum_{i=1}^{N} W_i(M)$$

Where $W_i(M)$ is the displacement at point M due to contact pressure p_i affected on an annulus with radius $\bar{r}_i$. With using the collocation method, the pressure p_i is constant on i^{th} element, and $W_i(M)$ is determined as follows:

$$W_i(M) = p_i \bar{W}_i(M) = p_i \int_{m_i}^{n_i} \bar{w}_h(r,z;r')\, dr',$$

$$m_i = \sum_{j=1}^{N} L_j - L_N,\ \ n_i = \sum_{j=1}^{N} L_j,\ (i = 1,2,3,\ldots,N).$$

4. Numerical results

In this part, some numerical evaluations are presented to illustrate the displacements and stresses at both the contact area and the half-space. The calculations consist of numerical evaluation of the integrals given in the equations of part 2.3. Detailed investigations of the integral show that some special attention is needed, due to the existence of poles and Bessel functions in the integrand. An adaptive procedure is needed due to the improper upper limit for the integrals.

For a flexible solid (full) plate under the tensionless-frictionless contact conditions, however, one must determine the radius α of the contact zone in the first stage. α, which makes the boundary value problem a nonlinear one, is determined by applying a trial and error procedure (see Fig 3.).

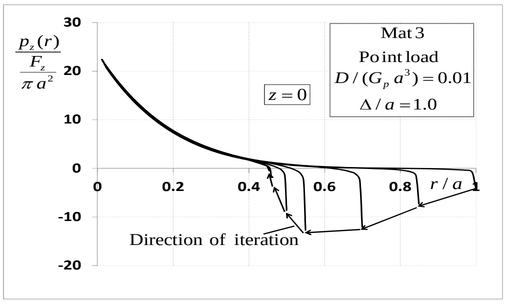

Fig 3. Iteration of traction solution for tensionless contact under a central point load (Mat 3).

5- Conclusions

In this study, a transversely isotropic half-space under the effect of a flexible circular foundation with axisymmetric distributed load has been investigated in detail. Proceeding to the main goal of this study, both the ring load and point load displacement Green's functions have been determined for the solid and annular plate where applicable. In addition, the ring load vertical displacement Green's function has been presented for the half-space. With the use of continuity of tractions and displacements at the contact area and with defining several inequalities, the governing equations for the tensionless-frictionless contact have been given. The nonlinear boundary value problem has been solved via a trial and error procedure combined with an integral equation and several inequalities. A ring-shape finite element with constant pressure (collocation method) has been implemented for numerical purposes. It has been shown that both the relative plate-subgrade stiffness and the degree of anisotropy of half-space affect the stress and displacement at any point of the plate and the half-space as well.